Forecast Combination and Reconciliation

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Who I am

I'm a Postdoctoral Researcher at the **University of Padova**

Research interests

- (Multivariate) economic time series (e.g. *energy*, *finance*, *macroeconomic*)
- Forecast combination and reconciliation
- Statistical software
  - `FoReco`: Forecast Reconciliation
  - `FoRecoPy`: Forecast Reconciliation in Python
  - `FoCo2` : Coherent Forecast Combination

Today's contributions

Joint work with Prof. Tommaso Di Fonzo

Bates and Granger (1969): linear forecast combination

+

Stone *et al.* (1942): constrained multivariate least-squares adjustment

↓

optimal combined and coherent forecasts

for multiple **linearly constrained** time series

Today's contributions

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optimal combined and coherent forecasts

for multiple **linearly constrained** time series

- New result that **unifies linear forecast reconciliation and combination** in a simultaneous and statistically justified way, improving accuracy and ensuring coherence of the forecasts

arXiv.2412.03429

Overview

Single-task forecast combination
(Bates and Granger, 1969; Timmermann, 2006) $\longrightarrow$ *multiple experts*, `no coherence`

Local: *one* variable $\uparrow$

Global: $n \geq 2$ variables $\downarrow$

Forecast reconciliation
(Stone *et al.*, 1942; Hyndman *et al.*, 2011) $\longrightarrow$ *single expert* and `coherence`

Overview

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Forecast reconciliation
(Stone *et al.*, 1942; Hyndman *et al.*, 2011) $\longrightarrow$ *single expert* and `coherence`

Multi-task forecast combination
(On the top of Sun and Deng, 2004 and Lavancier and Rochet, 2016) $\longrightarrow$ *multiple experts*, `no coherence`

Sequential coherent combination
Optimal coherent combination
$\longrightarrow$ *multiple experts* and `coherence`

`arXiv.2412.03429`

Linear forecast combination

Bates and Granger (1969); Timmermann (2006)

- Multiple forecasts of a single variable made by $p \geq 2$ different experts are combined to produce a new forecast
- Strengths and weaknesses of each expert (e.g., different models) are exploited to improve forecast accuracy

Linear forecast combination

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- Multiple forecasts of a single variable made by $p \geq 2$ different experts are combined to produce a new forecast
- Strengths and weaknesses of each expert (e.g., different models) are exploited to improve forecast accuracy
- **Single-task linear forecast combination**: $\widehat{y}_i^c = \boldsymbol{\omega}_i^\top \widehat{\boldsymbol{y}}_i = \sum_{j=1}^{p} \omega_{ij} \widehat{y}_i^j$

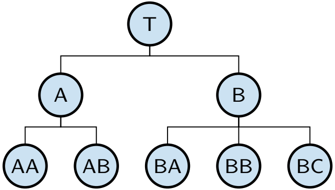
  $\widehat{\boldsymbol{y}}_i$ is the vector of p forecasts

  $\boldsymbol{\omega}_i \in \mathbb{R}^p$ is the vector of combination weights:

  - ew $\rightarrow$ equal weights
  - ow_{var} $\rightarrow$ optimal weights inversely proportional to MSE (Bates and Granger, 1969)
  - ow_{cov} $\rightarrow$ optimal weights in the unit simplex with MSE matrix (Conflitti *et al.*, 2015)

Forecast reconciliation

Athanasopoulos *et al.* (2024)



Constraints

$\mathrm{T} = \mathrm{A} + \mathrm{B}$

$\mathrm{A} = \mathrm{AA} + \mathrm{AB}$

$\mathrm{B} = \mathrm{BA} + \mathrm{BB} + \mathrm{BC}$

Genuine hierarchical time series

- Collection of n time series organized in a tree-like structure of aggregation
- The structure is nested, aggregation moves from bottom to upper levels

Forecast reconciliation

Athanasopoulos *et al.* (2024)

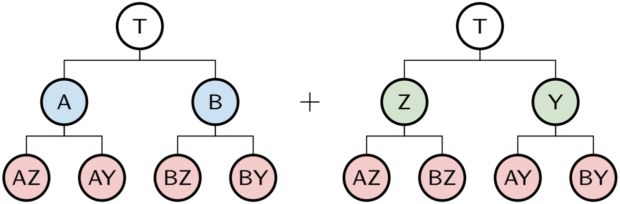
T	A	B
Z	AZ	BZ
Y	AY	BY

Constraints

$\mathrm{T} = \mathrm{A} + \mathrm{B}$
$\mathrm{A} = \mathrm{AZ} + \mathrm{AY}$
$\mathrm{B} = \mathrm{BZ} + \mathrm{BY}$
$\mathrm{T} = \mathrm{Z} + \mathrm{Y}$
$\mathrm{Z} = \mathrm{AZ} + \mathrm{BZ}$
$\mathrm{Y} = \mathrm{AY} + \mathrm{BY}$

Grouped time series

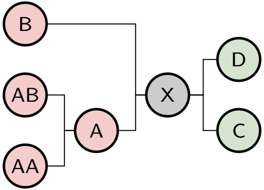
- Collection of n time series defined by cross-classifications rather than hierarchy
- Two or more genuine hierarchies sharing the same top and bottom variables



E.g.: sales by product × channel, tourism by region × purpose, or energy generation by fuel type × geography

Forecast reconciliation

Athanasopoulos *et al.* (2024)



Constraints

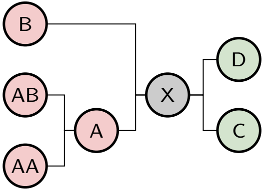
$$X = \mathrm{A} + \mathrm{B}$$
$$X = \mathrm{C} + \mathrm{D}$$
$$A = \mathrm{AA} + \mathrm{AB}$$

General linearly constrained time series

- Collection of n time series subjected to linear constraints (not just sums)
- Hierarchical and grouped structures are special cases of linearly constrained systems

Forecast reconciliation

Athanasopoulos *et al.* (2024)



Constraints

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General linearly constrained time series

- Collection of n time series subjected to linear constraints (not just sums)
- Hierarchical and grouped structures are special cases of linearly constrained systems

1. Forecast **all series at all levels** of aggregation $\rightarrow$ base forecasts (*single expert*)
2. Make the base forecasts **coherent** (post-forecasting process) $\rightarrow$ reconciled forecasts

Target	Base forecasts		Reconciled forecasts
$\boldsymbol{C}\boldsymbol{y} = \mathbf{0}$	$\boldsymbol{C}\widehat{\boldsymbol{y}} \neq \mathbf{0}$	$\rightarrow$	$\boldsymbol{C}\widetilde{\boldsymbol{y}} = \mathbf{0}$

Optimal linear forecast reconciliation

Athanasopoulos *et al.* (2024); Stone *et al.* (1942)

- **Projection approach** (zero-constrained representation)

$$\widehat{\boldsymbol{y}} = \boldsymbol{y} + \boldsymbol{\varepsilon} \text{ s.t. } \boldsymbol{C}\boldsymbol{y} = 0 \quad \Rightarrow \quad \min_{\boldsymbol{y}} \ (\widehat{\boldsymbol{y}} - \boldsymbol{y})^\top \boldsymbol{W}^{-1} (\widehat{\boldsymbol{y}} - \boldsymbol{y}) \text{ s.t. } \boldsymbol{C}\boldsymbol{y} = 0$$

$$\Rightarrow \quad \widetilde{\boldsymbol{y}} = \left[\boldsymbol{I} - \boldsymbol{W}\boldsymbol{C}^\top \left(\boldsymbol{C}\boldsymbol{W}\boldsymbol{C}^\top\right)^{-1} \boldsymbol{C}\right] \widehat{\boldsymbol{y}} = \boldsymbol{M}\widehat{\boldsymbol{y}}$$

- **In practice**, approximate forms of $\boldsymbol{W}$ are used, possibly using training set residuals

→ shrinkage approximation (Wickramasuriya *et al.*, 2019):

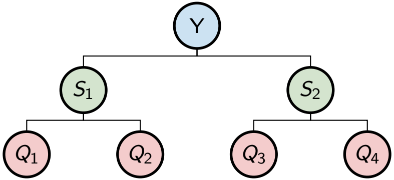
$$\boldsymbol{W} = \widehat{\lambda}\widehat{\boldsymbol{W}}_D + (1 - \widehat{\lambda})\widehat{\boldsymbol{W}}_1$$

$\widehat{\boldsymbol{W}}_1$ is the covariance matrix of the one-step ahead in-sample errors ($\widehat{\boldsymbol{e}}_t = \boldsymbol{y}_t - \widehat{\boldsymbol{y}}_t$),
$\widehat{\boldsymbol{W}}_D = \boldsymbol{I}_n \odot \widehat{\boldsymbol{W}}_1$, where $\odot$ denotes the Hadamard product

Temporal and cross-temporal frameworks

Athanasopoulos *et al.* (2017); Di Fonzo and Girolimetto (2023)

Temporal framework



Quarterly hierarchy:
quarterly, semi-annual and annual series

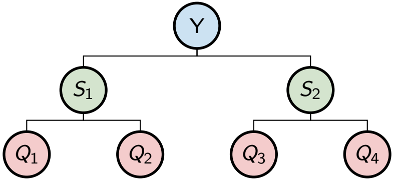
One variable observed at **different** frequencies

Non-overlapping aggregation (or *linear combination*) of the observations of a time series at regular intervals

Temporal and cross-temporal frameworks

Athanasopoulos *et al.* (2017); Di Fonzo and Girolimetto (2023)

Temporal framework

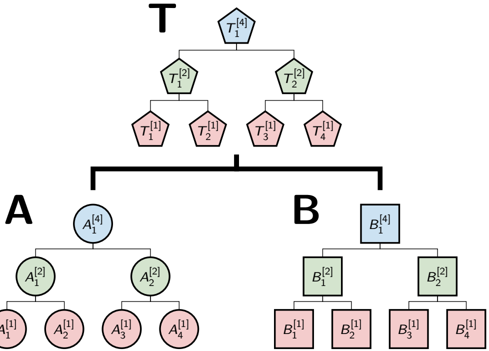


Quarterly hierarchy:
quarterly, semi-annual and annual series

One variable observed at **different** frequencies

Non-overlapping aggregation (or *linear combination*) of the observations of a time series at regular intervals

Cross-temporal framework



Linearly constrained time series observed at **different** frequencies

- **Starting points:**
  - Target vector $\boldsymbol{y} \in \mathbb{R}^n$ s.t. $\boldsymbol{C}\boldsymbol{y} = \boldsymbol{0}_{(n_u \times 1)}$
  - Base forecasts of the n individual variables made by $p \geq 2$ experts:

$$\widehat{\boldsymbol{y}}^1 \in \mathbb{R}^{n_1}, \; \dots, \; \widehat{\boldsymbol{y}}^p \in \mathbb{R}^{n_p} \qquad 1 \leq n_j \leq n, \quad j = 1, \dots, p$$

- **Unbalanced case**: the forecasts provided by each expert might refer to **different sets of individual variables** $\rightarrow$ $(m = \sum_{j=1}^{p} n_j)$

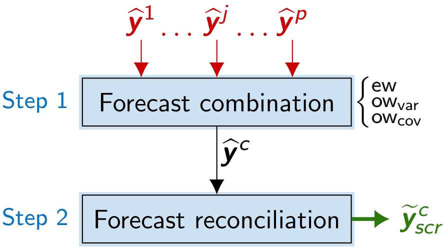
- *Selection matrix*: $\boldsymbol{L} = \text{Diag}\left(\boldsymbol{L}_1, \dots, \boldsymbol{L}_p\right) \in \{0,1\}^{m \times np}$, where $\boldsymbol{L}_j \in \{0,1\}^{n_j \times n}$ selects the $n_j \leq n$ entries of $\boldsymbol{y}$ for which base forecasts of the j-th expert are available

- **Balanced case**: $n_j = n \quad \Rightarrow \quad \boldsymbol{L} = \boldsymbol{I}_m, \; m = np$

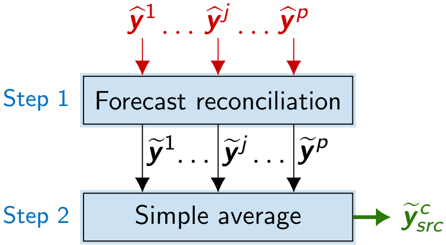
Sequential coherent forecast reconciliation

Girolimetto and Di Fonzo (2024b)

combination + reconciliation



reconciliation + simple average



- ✗ src approach is limited to the balanced case and does not apply to more general situations
- In the following, we consider scr_{var} and scr_{cov}, with ow_{var} and ow_{cov}, respectively
- In the working paper, we discuss also about scr_{ew} and src

Optimal coherent forecast combination

Girolimetto and Di Fonzo (2024b): n variables, p experts

- Assuming unbiased base forecasts,

$$\widehat{y}_i^j = y_i + \varepsilon_i^j, \quad i = 1, \dots, n, \quad j = 1, \dots, p$$

- Linear relationship linking $\widehat{\boldsymbol{y}}$ and $\boldsymbol{y}$: as

$$\widehat{\boldsymbol{y}} = \begin{bmatrix} \widehat{\boldsymbol{y}}^1 \\ \vdots \\ \widehat{\boldsymbol{y}}^p \end{bmatrix} = \boldsymbol{K}\boldsymbol{y} + \boldsymbol{\varepsilon}, \quad \text{s.t.} \quad \boldsymbol{C}\boldsymbol{y} = \boldsymbol{0}_{(n_u \times 1)}$$

  where $\boldsymbol{K} = \boldsymbol{L}\left(\boldsymbol{1}_p \otimes \boldsymbol{I}_n\right) \in \{0, 1\}^{m \times np}$, and $\boldsymbol{\varepsilon}$ is a zero-mean random vector with $(m \times m)$ covariance matrix $\boldsymbol{W} = E(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^\top)$

- **Linearly constrained quadratic program**:

$$\widetilde{\boldsymbol{y}}^c = \arg\min_{\boldsymbol{y}} \left(\widehat{\boldsymbol{y}} - \boldsymbol{K}\boldsymbol{y}\right)^\top \boldsymbol{W}^{-1} \left(\widehat{\boldsymbol{y}} - \boldsymbol{K}\boldsymbol{y}\right) \qquad \text{s.t.} \;\; \boldsymbol{C}\boldsymbol{y} = \boldsymbol{0}_{(n_u \times 1)}$$

MMSE linear coherent combined forecasts

arXiv.2412.03429

The minimum mean square error (MMSE) linear coherent combined forecast vector is given by

$$\widetilde{\boldsymbol{y}}^c = \boldsymbol{M} \; \boldsymbol{\Omega}^\top \; \widehat{\boldsymbol{y}}$$

with weight matrix $\boldsymbol{\Psi}^\top = \boldsymbol{M}\boldsymbol{\Omega}^\top \in \mathbb{R}^{n \times m}$ and error covariance matrix $\widetilde{\boldsymbol{W}}_c = \boldsymbol{M}\boldsymbol{W}_c$, where

Property: $\widetilde{\boldsymbol{y}}^c$ is unbiased and $\boldsymbol{L}_j \widetilde{\boldsymbol{W}}_c \boldsymbol{L}_j^\top \preceq \boldsymbol{L}_j \boldsymbol{W}_c \boldsymbol{L}_j^\top \preceq \boldsymbol{W}_j$, $j = 1, \dots, p$

MMSE linear coherent combined forecasts

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The minimum mean square error (MMSE) linear coherent combined forecast vector is given by

$$\widetilde{\boldsymbol{y}}^c = \boldsymbol{M} \underbrace{\boldsymbol{\Omega}^\top}_{\text{maps } \widehat{\boldsymbol{y}} \in \mathbb{R}^m \to \widehat{\boldsymbol{y}}^c \in \mathbb{R}^n} \widehat{\boldsymbol{y}}$$

with weight matrix $\boldsymbol{\Psi}^\top = \boldsymbol{M}\boldsymbol{\Omega}^\top \in \mathbb{R}^{n \times m}$ and error covariance matrix $\widetilde{\boldsymbol{W}}_c = \boldsymbol{M}\boldsymbol{W}_c$, where

$$\boldsymbol{\Omega} = \boldsymbol{W}^{-1}\boldsymbol{K}\boldsymbol{W}_c, \qquad \boldsymbol{W}_c = \left(\boldsymbol{K}^\top \boldsymbol{W}^{-1} \boldsymbol{K}\right)^{-1}$$

Property: $\widetilde{\boldsymbol{y}}^c$ is unbiased and $\boldsymbol{L}_j \widetilde{\boldsymbol{W}}_c \boldsymbol{L}_j^\top \preceq \boldsymbol{L}_j \boldsymbol{W}_c \boldsymbol{L}_j^\top \preceq \boldsymbol{W}_j$, $j = 1, \dots, p$

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The minimum mean square error (MMSE) linear coherent combined forecast vector is given by

$$\widetilde{\boldsymbol{y}}^c = \underbrace{\boldsymbol{M}}_{\text{projects } \widehat{\boldsymbol{y}}^c \to \mathcal{S} = \{\boldsymbol{y} \in \mathbb{R}^n \mid \boldsymbol{C}\boldsymbol{y} = \boldsymbol{0}_{(n_u \times 1)}\}} \overbrace{\boldsymbol{\Omega}^\top}^{\text{maps } \widehat{\boldsymbol{y}} \in \mathbb{R}^m \to \widehat{\boldsymbol{y}}^c \in \mathbb{R}^n} \widehat{\boldsymbol{y}}$$

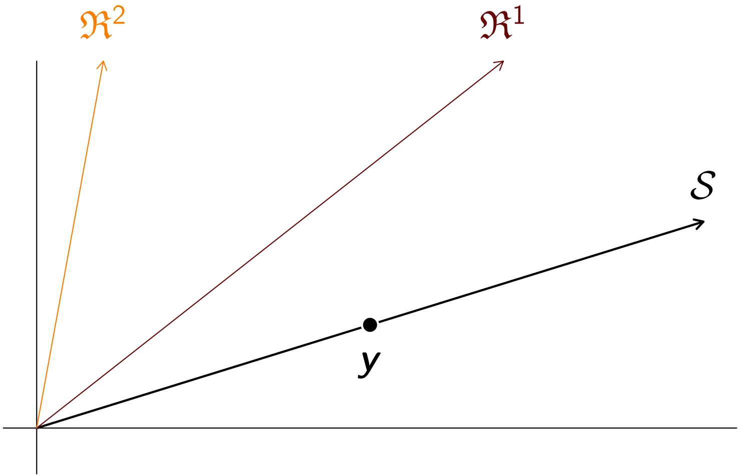
with weight matrix $\boldsymbol{\Psi}^\top = \boldsymbol{M}\boldsymbol{\Omega}^\top \in \mathbb{R}^{n \times m}$ and error covariance matrix $\widetilde{\boldsymbol{W}}_c = \boldsymbol{M}\boldsymbol{W}_c$, where

$$\boldsymbol{\Omega} = \boldsymbol{W}^{-1}\boldsymbol{K}\boldsymbol{W}_c, \qquad \boldsymbol{W}_c = \left(\boldsymbol{K}^\top \boldsymbol{W}^{-1}\boldsymbol{K}\right)^{-1} \qquad \boldsymbol{M} = \left[\boldsymbol{I}_n - \boldsymbol{W}_c\boldsymbol{C}^\top \left(\boldsymbol{C}\boldsymbol{W}_c\boldsymbol{C}^\top\right)^{-1}\boldsymbol{C}\right]$$

Property: $\widetilde{\boldsymbol{y}}^c$ is unbiased and $\boldsymbol{L}_j\widetilde{\boldsymbol{W}}_c\boldsymbol{L}_j^\top \preceq \boldsymbol{L}_j\boldsymbol{W}_c\boldsymbol{L}_j^\top \preceq \boldsymbol{W}_j$, $j = 1, \dots, p$

Graphical visualisation

2 experts, n variables

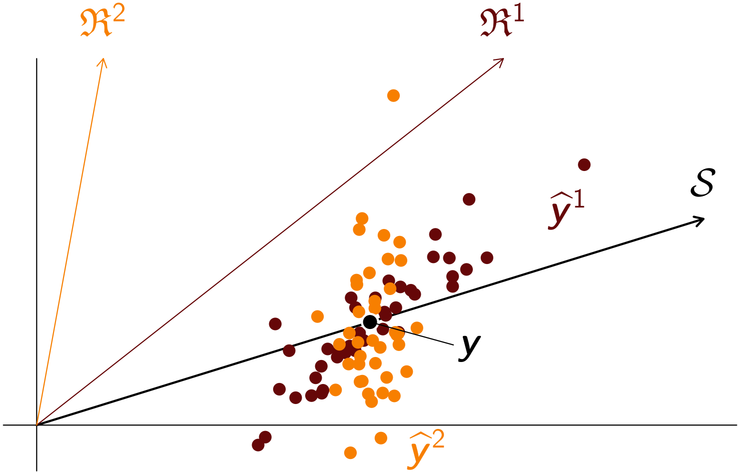


1. $\mathfrak{R}^1$ and $\mathfrak{R}^2$ show the most likely direction of deviations from the coherent subspace $\mathcal{S}$ for the 2 experts.
   The black dot $\boldsymbol{y}$ denotes the (unknown) target forecast.

$$\widetilde{\boldsymbol{y}}^c = \boldsymbol{M} \, \boldsymbol{\Omega}^\top \, \widehat{\boldsymbol{y}}$$

Graphical visualisation

2 experts, n variables



2. **Red** and **orange** points indicate the potential base forecasts for the 2 experts, $\widehat{\boldsymbol{y}}^1$ and $\widehat{\boldsymbol{y}}^2$, respectively

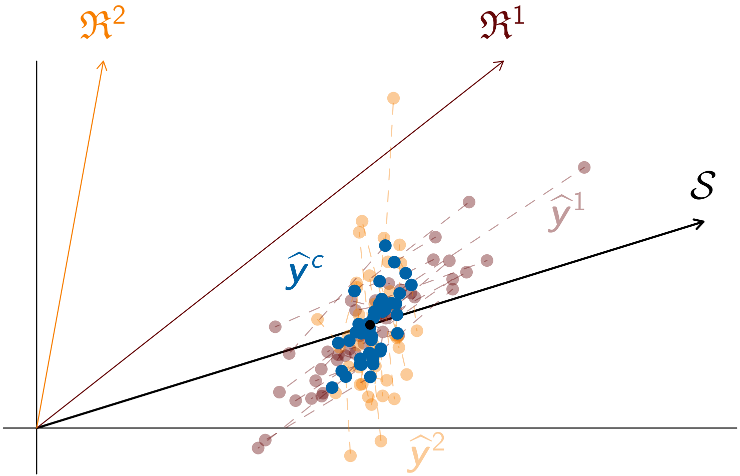
$$\Downarrow$$

$$\widehat{\boldsymbol{y}} = \begin{bmatrix} \widehat{\boldsymbol{y}}^1 \\ \widehat{\boldsymbol{y}}^2 \end{bmatrix}$$

$$\widetilde{\boldsymbol{y}}^c = M \, \Omega^\top \, \widehat{\boldsymbol{y}}$$

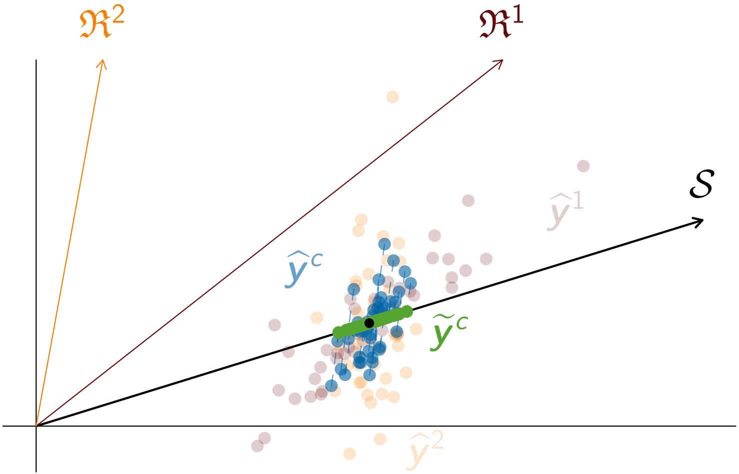
Graphical visualisation

2 experts, n variables



3. **Blue** points represent the unbiased MMSE linear multi-task combined forecasts, $\widehat{\boldsymbol{y}}^c = \boldsymbol{\Omega}^\top \widehat{\boldsymbol{y}}$.

$$\widetilde{y}^c = M \; \boldsymbol{\Omega}^\top \; \widehat{\boldsymbol{y}}$$



4. **Green** points represent the unbiased MMSE linear coherent combined forecasts, $\widetilde{\boldsymbol{y}}^c = \boldsymbol{M}\boldsymbol{\Omega}^\top\widehat{\boldsymbol{y}}$, as an oblique projection of $\widehat{\boldsymbol{y}}^c$ on $\mathcal{S}$.

$$\widetilde{\boldsymbol{y}}^c = \boldsymbol{M}\ \boldsymbol{\Omega}^\top\ \widehat{\boldsymbol{y}}$$

About the covariance matrix

arXiv.2412.03429

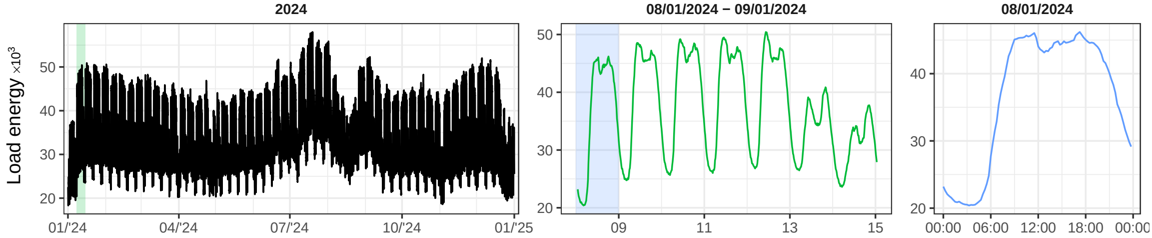
- Matrix $\boldsymbol{W}$ determines **how the base forecasts are combined**, and then the **nature** of the coherent forecasts
- Special case: *by-expert* **block-diagonal** shrunk error covariance matrix (occ)

$$\boldsymbol{W} = \begin{bmatrix} \widehat{\boldsymbol{W}}_{1,shr} & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & \widehat{\boldsymbol{W}}_{j,shr} & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & \widehat{\boldsymbol{W}}_{p,shr} \end{bmatrix} \xrightarrow{j=1,\dots,p} \begin{cases} \widehat{\boldsymbol{W}}_{j,shr} = \widehat{\lambda}_j \left(\boldsymbol{I}_n \odot \widehat{\boldsymbol{W}}_j\right) + \left(1 - \widehat{\lambda}_j\right) \widehat{\boldsymbol{W}}_j \\ \widehat{\boldsymbol{W}}_j = \dfrac{1}{T}\displaystyle\sum_{t=1}^{T} \widehat{\boldsymbol{\varepsilon}}_t^j \widehat{\boldsymbol{\varepsilon}}_t^{j\top} \\ \widehat{\lambda}_j \rightarrow \text{Schäfer and Strimmer (2005)} \end{cases}$$

- **How to estimate $\boldsymbol{W}$:**
  1. in-sample errors (reconciliation) ➔ Australian electricity generation dataset
  2. validation errors (combination) ➔ **Italian energy load dataset by Terna**

Italian energy load forecasting by Terna

Terna is the Europe's largest independent electricity Transmission System Operator (TSO)



- **Terna** processes the official statistics of the entire national electricity sector and is responsible for official communications to international bodies such as Eurostat, UN, . . .
- Among the various activities, Terna currently **publishes** on its data portal very short-term load forecasts for the next day, at national level and disaggregated by 7 bidding zones
- Historical 15-minutes time series of observed and forecast load may be **easily downloaded**

The forecasting experiment

15-min data: rolling forecast experiment with daily iterations (2024) and 96-step ahead forecast horizons

- 8 variables → Italy + 7 Bidding Zones (BZ)
- **Range**: 1/1/2023 – 31/12/2024 with 365 days as validation set to compute optimal weights and error covariance matrices
- **Test set**: all the 366 days of 2024
- **Accuracy evaluation**: AvgRelMAE (geometric Average Relative Mean Absolute Error) and DM-test
- **Coherency issue**: The aggregated forecasts for the 7 BZ must match the forecasts for Italy

Forecast and combination approaches

R package: `FoCo2` (Girolimetto and Di Fonzo, 2024a)

Base forecasts:

- Terna, exploiting a comprehensive set of influencing factors, including meteorological data, climate trends, and socio-economic variables
- the daily random walk (drw): $\widehat{y}_{i,t+h|t} = y_{i,t-96+h}$

Local-single-task combination procedures

- Equal weights (ew)
- **Optimal single-task combination** (ow_{var} and ow_{cov})

Global-multi-task combination procedures

- **Sequential local-combination-then-reconciliation** (scr_{var} and scr_{cov})
- **Optimal multi-task combination** (occ) using a *by-expert* **block-diagonal** shrunk error covariance matrix

Legend: incoherent / coherent forecasts

AvgRelMAE

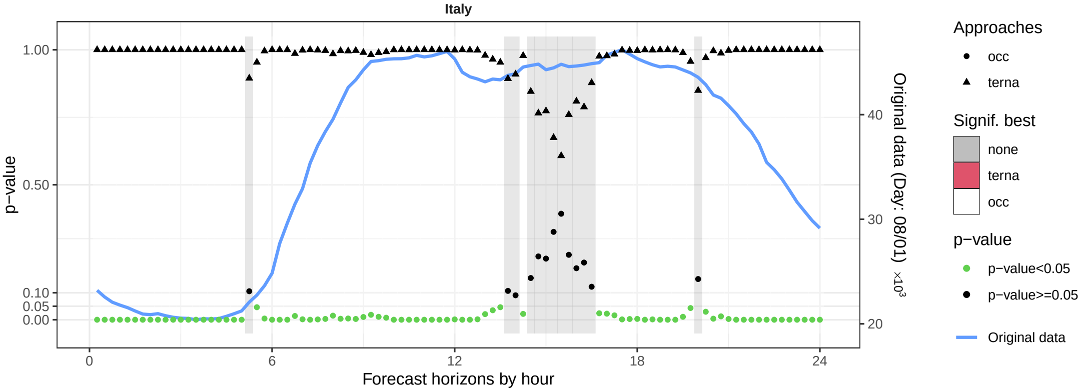
Bold entries identify the best approach. Red denotes approaches worse then Terna (benchmark)

	Country and 7 bidding zones									
App.	Italy	North	C-North	C-South	South	Calabria	Sicily	Sardinia	**BTS**	**All**
drw	4.6781	5.7847	5.1689	4.4872	6.0555	4.5870	3.1265	2.2250	4.2710	4.3199
ew	2.5376	3.0746	2.7368	2.3780	3.1048	2.4001	1.7056	1.2877	2.2872	2.3171
ow_{var}	0.9930	0.9980	0.9909	0.9897	0.9943	0.9879	0.9663	0.9282	0.9791	0.9808
ow_{cov}	0.9863	0.9905	0.9847	0.9847	0.9930	0.9854	0.9676	0.9312	0.9765	0.9777
scr_{var}	0.9863	0.9977	0.9889	0.9881	0.9927	0.9867	0.9648	0.9267	0.9777	0.9787
scr_{cov}	0.9827	0.9911	0.9841	0.9844	0.9926	0.9848	0.9674	0.9309	0.9763	0.9771
occ	**0.8973**	**0.8997**	**0.8969**	**0.8966**	**0.8952**	**0.8947**	**0.8936**	**0.8936**	**0.8958**	**0.8960**

- When using the global approaches, either two-step or optimal, more accurate forecasts are obtained
- **occ** approach consistently outperforms **Terna** and **all the other combinations**

Diebold-Mariano tests for each 15-min forecast horizon

Terna *vs* occ forecasts – 96 different forecast horizons – absolute loss – Italy



- occ **significantly** outperforms Terna in $\sim 86\%$ of the cases, with no improvements between $1-4$ pm
- Terna never significantly improve with respect to occ

Conclusions

- We propose a novel method to address the challenge of **combining forecasts** from **multiple** experts for **linearly constrained** time series. This method ensures **coherent** forecasts
- We show that a coherent combination approach produces significantly more accurate forecasts **immediately** after Terna publishes the previous day's energy load and the current day's forecasts on its data portal
- In the working paper, we expand on today's presentation with simulations and an additional real-world application on Australian daily electricity generation

$\Downarrow$

The **optimal coherent combination** almost always provides the most accurate forecasts

- **Future research**: investigate the roles of $\boldsymbol{M}$, $\boldsymbol{\Omega}$ and $\boldsymbol{W}$ in the MMSE formula, and extend the framework to temporal, cross-temporal and probabilistic forecasting

Coherent forecast combination for linearly constrained multiple time series

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Abstract

Linearly constrained multiple time series may be encountered in many practical contexts, such as the National Accounts (e.g., GDP disaggregated by Income, Expenditure and Output), and multilevel frameworks where the variables are organized according to hierarchies or groupings, like the total energy consumption of a country disaggregated by region and energy sources. In these cases, when multiple incoherent base forecasts for each individual variable are available, a forecast combination-and-reconciliation approach, that we call *coherent forecast combination*, may be used to improve the accuracy of the base forecasts and achieve coherence in the final result. In this paper, we develop an optimization-based technique that combines multiple unbiased base forecasts while assuring the constraints valid for the series. We present closed form expressions for the coherent combined forecast vector and its error covariance matrix in the general case where a different number of forecasts is available for each variable. We also discuss practical issues related to the covariance matrix that is part of the optimal solution. Through simulations and a forecasting experiment on the daily Australian electricity generation hierarchical time series, we show that the proposed methodology, in addition to adhering to sound statistical principles, may yield in significant improvement on base forecasts, single-task combination and single-expert reconciliation approaches as well.

Keywords: Forecasting; Linearly constrained multiple time series; Coherent forecasts; Forecast combination; Forecast reconciliation; Australian electricity generation

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THANK YOU!

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danigiro/FoCo2

CRAN/FoCo2

FoCo²

Quantitative Economics seminar, Maastricht University

Maastricht, Netherlands – 01/10/2025

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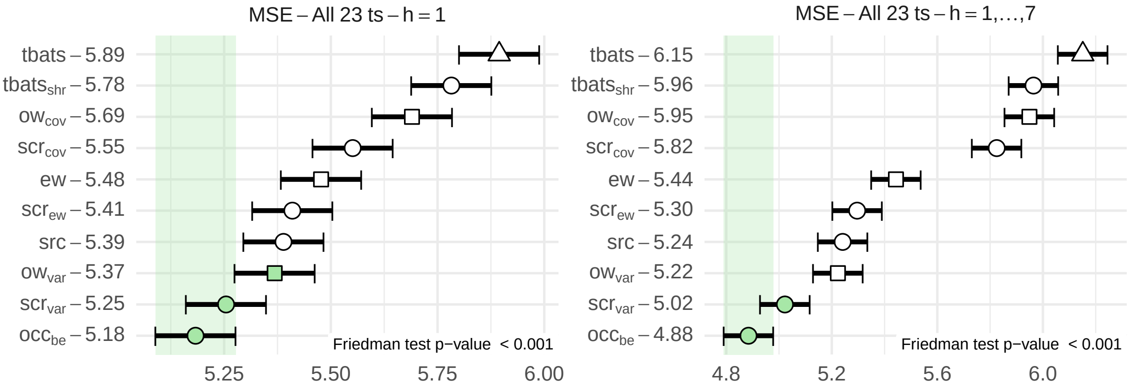
AvgRelMAE for the Australian electricity generation dataset

Red: worse than the benchmark (ew). Bold: the best approach. Italic: second best approach

Approach	**Forecast horizon** $h=1$	$h=2$	$h=3$	$h=4$	$h=5$	$h=6$	$h=7$	1:7
Base (incoherent forecasts) and single model reconciliation								
tbats	1.0796	1.0780	1.0445	1.0270	1.0322	1.0288	1.0142	1.0393
$\text{tbats}_{\text{shr}}$	1.0478	1.0577	1.0304	1.0108	1.0219	1.0213	1.0116	1.0257
Combination (incoherent forecasts)								
ew	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
ow_{var}	0.9840	0.9881	0.9995	1.0032	1.0020	1.0028	1.0054	0.9995
ow_{cov}	1.0279	1.0494	1.0972	1.1103	1.1009	1.0993	1.1055	1.0908
Coherent combination								
src	0.9827	0.9855	0.9863	*0.9833*	**0.9852**	**0.9873**	**0.9911**	*0.9859*
scr_{ew}	0.9875	0.9898	0.9859	0.9859	*0.9885*	*0.9905*	*0.9962*	0.9890
scr_{var}	*0.9586*	*0.9683*	*0.9838*	0.9942	0.9982	1.0017	1.0114	0.9910
scr_{cov}	1.0026	1.0287	1.0795	1.0972	1.0942	1.0913	1.0981	1.0773
occ	**0.9481**	**0.9560**	**0.9754**	**0.9831**	0.9891	0.9939	0.9993	**0.9808**

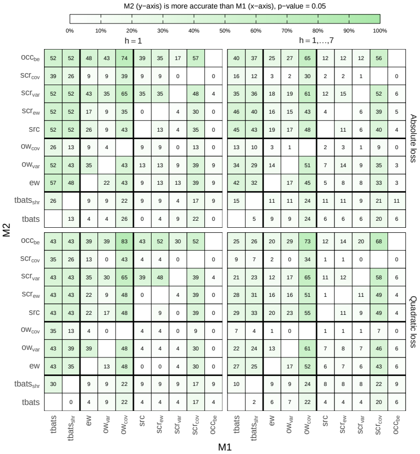
MCB Nemenyi test

R package tsutils (Kourentzes, 2023). The Friedman test p-value is reported in the lower right corner. The mean rank of each approach is shown to the right of its name. Statistical differences are indicated if the intervals of two forecast procedures do not overlap



Diebold and Mariano (1995) test

Pairwise DM-test results evaluated using absolute loss (top panels) and quadratic loss (bottom panel) across different forecast horizons. The left panel corresponds to forecast horizon $h = 1$, while the right panel is for $h = 1, \dots, 7$. Each cell reports the percentage of series for which the p-value of the DM-test is below 0.05



Model Confidence Set

MCS results (10^4 bootstrap sample) evaluated using absolute loss (top panels) and quadratic loss (bottom panel) across different forecast horizons ($h = 1$ and $h = 1, \dots, 7$). Each cell reports the percentage of series for which that approach is in the Model Confidence Set across different thresholds ($\delta \in \{95\%, 90\%, 80\%\}$)

	$h=1$			$h=1:7$		
Approach	$\delta=95\%$	$\delta=90\%$	$\delta=80\%$	$\delta=95\%$	$\delta=90\%$	$\delta=80\%$
Absolute loss - All 23 time series						
Base (incoherent forecasts) and single model reconciliation						
tbats	56.5	56.5	52.2	78.3	69.6	56.5
$\text{tbats}_{\text{shr}}$	78.3	73.9	60.9	87.0	82.6	69.6
Combination (incoherent forecasts)						
ew	87.0	87.0	78.3	**95.7**	*91.3*	78.3
ow_{var}	*95.7*	*95.7*	82.6	**95.7**	*91.3*	*82.6*
ow_{cov}	73.9	69.6	60.9	73.9	65.2	43.5
Coherent combination						
src	91.3	91.3	87.0	**95.7**	**95.7**	**87.0**
scr_{ew}	91.3	91.3	87.0	**95.7**	*91.3*	78.3
scr_{var}	**100.0**	**100.0**	*91.3*	*91.3*	*91.3*	**87.0**
scr_{cov}	82.6	78.3	73.9	78.3	69.6	65.2
occ	**100.0**	**100.0**	**95.7**	**95.7**	**95.7**	**87.0**

	$h=1$			$h=1:7$		
Approach	$\delta=95\%$	$\delta=90\%$	$\delta=80\%$	$\delta=95\%$	$\delta=90\%$	$\delta=80\%$
Quadratic loss - All 23 time series						
Base (incoherent forecasts) and single model reconciliation						
tbats	65.2	65.2	60.9	*91.3*	73.9	73.9
$\text{tbats}_{\text{shr}}$	73.9	69.6	65.2	**95.7**	82.6	69.6
Combination (incoherent forecasts)						
ew	87.0	78.3	60.9	**95.7**	*87.0*	*82.6*
ow_{var}	**100.0**	82.6	78.3	*91.3*	*87.0*	*82.6*
ow_{cov}	78.3	69.6	52.2	78.3	60.9	34.8
Coherent combination						
src	*95.7*	*91.3*	*82.6*	*91.3*	**91.3**	**91.3**
scr_{ew}	*95.7*	87.0	73.9	*91.3*	**91.3**	**91.3**
scr_{var}	*95.7*	**95.7**	**95.7**	*91.3*	**91.3**	**91.3**
scr_{cov}	82.6	82.6	73.9	73.9	60.9	56.5
occ	**100.0**	**95.7**	**95.7**	*91.3*	**91.3**	**91.3**

Australian electricity generation dataset

Daily electricity generation from various energy sources in Australia (AEMO, Panagiotelis *et al.*, 2023)

Linear combination matrix (8×15)

−1 0 1

	Wind	Biomass	Battery (Discharg.)	Battery (Charg.)	Hydro	Pumps	Solar (Rooftop)	Solar (Utility)	Distillate	Black Coal	Brown Coal	Gas (Recip.)	Gas (OCGT)	Gas (CCGT)	Gas (Steam)
Total	+	+	+	−	+	−	+	+	+	+	+	+	+	+	+
non−Renew.									+	+	+	+	+	+	+
Renew.	+	+	+	−	+	−	+	+							
Batteries			+	−											
Hydro (−Pumps)					+	−									
Solar							+	+							
Coal										+	+				
Gas												+	+	+	+

23 time series with 15 bottom-level series

- **Range**: 11 June 2019 – 10/06/2020 (1 year)
- Forecasting experiment: expanding window, daily step and 7-step ahead forecast horizons
- $p = 3$ base forecasts (R package `forecast`):
  - stlf Seasonal and Trend decomposition using Loess
  - arima AutoRegressive Integrated Moving Average
  - tbats Exponential smoothing with Box-Cox transformation, ARMA errors, Trend and Seasonality
- **Accuracy indices**: AvgRelMAE and AvgRelMSE + MCB, MCS and DM-test
- **Negativity issues**: numerical optimization with non-negativity and equality constraints

AvgRelMAE for the Australian electricity generation dataset

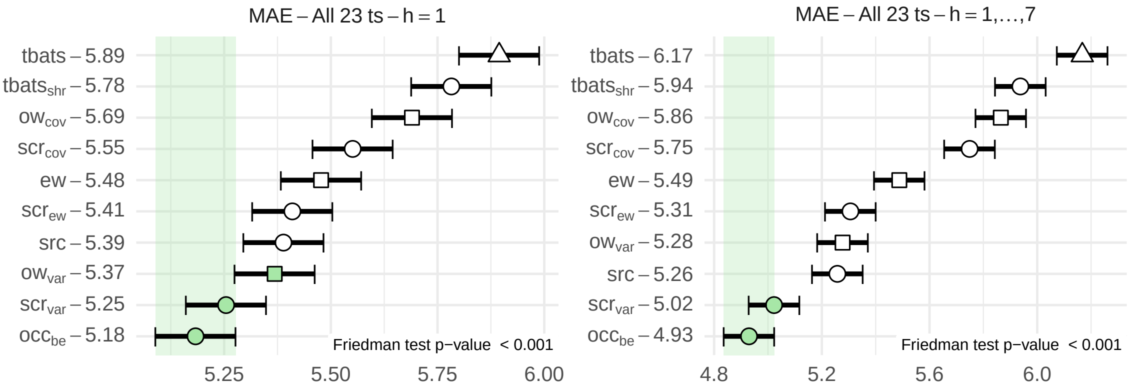
Red: worse than the benchmark (ew). Bold: the best approach. Italic: second best approach

Approach	Forecast horizon $h=1$	$h=2$	$h=3$	$h=4$	$h=5$	$h=6$	$h=7$	1:7
Base (incoherent forecasts) and single model reconciliation								
tbats	1.0447	1.0515	1.0348	1.0266	1.0305	1.0288	1.0201	1.0331
$\text{tbats}_{\text{shr}}$	1.0320	1.0413	1.0231	1.0134	1.0212	1.0208	1.0188	1.0235
Combination (incoherent forecasts)								
ew	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
ow_{var}	0.9927	0.9921	0.9982	0.9983	0.9967	0.9967	0.9990	0.9965
ow_{cov}	1.0216	1.0208	1.0390	1.0423	1.0307	1.0250	1.0325	1.0309
Coherent combination								
src	0.9939	0.9941	0.9919	0.9895	0.9887	*0.9908*	*0.9933*	0.9915
scr_{ew}	0.9952	0.9959	0.9911	0.9908	0.9908	0.9932	0.9961	0.9930
scr_{var}	*0.9819*	*0.9803*	*0.9869*	*0.9895*	*0.9887*	0.9913	0.9972	*0.9882*
scr_{cov}	1.0081	1.0081	1.0270	1.0327	1.0245	1.0197	1.0250	1.0215
occ	**0.9779**	**0.9745**	**0.9843**	**0.9852**	**0.9851**	**0.9880**	**0.9926**	**0.9843**

- R package FoCo2
- Coherent forecast combination outperforms incoherent approaches
- scr_{var} is a strong alternative among sequential coherent combination procedures
- occ **provides the most accurate forecasts for all the horizons**

MCB Nemenyi test

R package `tsutils` (Kourentzes, 2023). The Friedman test *p*-value is reported in the lower right corner. The mean rank of each approach is shown to the right of its name. Statistical differences are indicated if the intervals of two forecast procedures do not overlap



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